

# Non-singular affine surfaces with self-maps

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## Abstract

We classify surjective self-maps (of degree at least two) of affine surfaces according to the log Kodaira dimension.

## §1. Introduction.

In this paper we are interested in the following question.

**Question.** Classify all smooth affine surfaces  $X/\mathbf{C}$  which admit a proper morphism  $f : X \rightarrow X$  with degree  $f > 1$ .

In [5] and [18], a classification of smooth projective surfaces with a self-map of degree  $> 1$  has been given. This paper is inspired by their results. The case when  $X$  is singular appears to be quite hard so we restrict ourselves to the smooth case. Similarly, if  $f$  is not a proper morphism then again the problem is difficult. For example, we do not even know if there is an étale map of degree  $> 1$  from  $\mathbf{C}^2$  to itself. This is the famous Jacobian Problem. If  $S$  is any  $\mathbf{Q}$ -homology plane with  $\bar{\kappa}(S) = -\infty$  then  $S$  admits an algebraic action of the additive group  $G_a$  ([14]). Hence the automorphism group of such a surface is infinite. However, the problem of constructing a proper self-map of degree  $> 1$  for  $S$  is quite non-trivial. Our main result can be stated as follows.

**Theorem.** *There is a complete classification of smooth complex affine surfaces  $X$  which admit a proper self-morphism of degree  $> 1$ , if either the logarithmic Kodaira dimension  $\bar{\kappa}(X) \geq 0$  or the topological fundamental group  $\pi_1(X)$  is infinite.*

More precisely, any such  $X$  is isomorphic to a quotient of the form  $(\Delta \times \mathbf{A}^1)/G$  or  $(\Delta \times \mathbf{C}^*)/G$  where  $\Delta$  is a smooth curve and  $G$  is a finite group acting freely on  $\Delta \times \mathbf{A}^1$  or  $\Delta \times \mathbf{C}^*$  respectively.

As a consequence of the proof, we have:

**Corollary.** Suppose that  $X$  is an affine surface with a proper morphism  $X \rightarrow B$  of degree  $> 1$ . Then we have:

- (1) If  $\bar{\kappa}(X) \geq 0$ , then  $\bar{\kappa}(X) = 0, 1$  and  $X \cong (\Delta \times \mathbf{C}^*)/G$  where  $\Delta$  is a smooth affine curve and  $G$  is a finite group acting freely on  $\Delta \times \mathbf{C}^*$ .
- (2) Suppose that  $\bar{\kappa}(X) = -\infty$  and let  $\varphi : X \rightarrow B$  be an  $\mathbf{A}^1$ -fibration (for the existence see [15], Chapter I, §3).
- (2a) If  $\bar{\kappa}(B) = -\infty$ , then the topological fundamental group  $\pi_1(X)$  is finite.
- (2b) If  $\bar{\kappa}(B) = 0$ , then  $B \cong \mathbf{C}^*$  and  $X \cong \mathbf{A}^1 \times \mathbf{C}^*$ .
- (2c) If  $\bar{\kappa}(B) = 1$  then every fibre of  $\varphi$  is reduced and irreducible and  $X \cong (\Delta \times \mathbf{A}^1)/(\mathbf{Z}/(m))$  ( $m \geq 1$ ) where  $\Delta$  is a smooth affine curve and  $\mathbf{Z}/(m)$  acts freely on  $\Delta \times \mathbf{A}^1$ .

*Remark.*

(1) There are easy examples of  $X$  where  $\bar{\kappa}(X) \geq 0$  or  $\pi_1(X)$  is infinite, but  $X$  has no proper self map of degree  $> 1$ . For example, let  $C, D$  be smooth irreducible projective curves of genus  $m, n \geq 2$  which admit no non-trivial automorphisms. Assume that  $m \neq n$  and let  $C', D'$  be non-empty affine open subsets of  $C, D$  respectively. Then  $C', D'$  have no proper self-maps of degree  $> 1$  and hence  $X := C' \times D'$  has no proper self-map of degree  $> 1$ . Clearly,  $\bar{\kappa}(X) = 2$  and  $\pi_1(X)$  is infinite.

It is an interesting problem to determine those quotients  $(\Delta \times \mathbf{A}^1)/G$  and  $(\Delta \times \mathbf{C}^*)/G$  which have no proper self maps of degree  $> 1$ .

(2) So far the authors have not been able to find any example (apart from  $\mathbf{C}^2$ ) of a smooth affine surface  $X$  with  $\bar{\kappa}(X) = -\infty$  and  $\pi_1(X)$  finite such that  $X$  has a proper self-map of degree  $> 1$ .

(3) Our proof of the theorem uses the classification theory of open algebraic surfaces developed by S. Iitaka, Y. Kawamata, T. Fujita, M. Miyanishi,

T. Sugie and other Japanese mathematicians. We also use topological arguments in an essential way.

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## §2. Preliminaries.

We will only deal with complex algebraic varieties. By a curve (resp. surface) we mean an irreducible, quasi-projective curve (resp. an irreducible, quasi-projective surface). By a *component* of a variety  $Z$  we mean an irreducible component of  $Z$ . Let  $Z$  be a smooth surface. By a  $\mathbf{P}^1$ -fibration on  $Z$  we mean a morphism  $Z \rightarrow B$  onto a smooth algebraic curve whose general fiber is isomorphic to  $\mathbf{P}^1$ . Similarly an  $\mathbf{A}^1$ -fibration and a  $\mathbf{C}^*$ -fibration on  $Z$  can be defined. Here,  $\mathbf{C}^*$  denotes  $\mathbf{P}^1 - \{\text{two points}\}$ .

Given a fibration  $\varphi : X \rightarrow B$  from a smooth surface  $X$  onto a smooth curve  $B$  let  $F := \sum m_i F_i$  be a scheme-theoretic fiber of  $\varphi$  where  $F_i$  are the irreducible components of  $F$ . The greatest common divisor of the integers  $m_i$  is called the *multiplicity* of  $F$ . If the multiplicity is  $> 1$  then we say that  $F$  is a *multiple fiber*.

A smooth projective irreducible rational curve  $C$  on a smooth surface  $Z$  with  $C^2 = n$  is called an  $(n)$ -curve. The topological Euler-Poincaré characteristic of a variety  $Z$  is denoted by  $\chi(Z)$ .

Given a smooth surface  $Z$  there is an open embedding  $Z \subset W$  such that  $W$  is a smooth projective surface and  $D := W \setminus Z$  is a divisor with simple normal crossings. If any  $(-1)$ -curve in  $D$  meets at least three other components of  $D$  then we say that  $W$  is a *good* compactification of  $Z$ .

For a smooth, irreducible, quasi-projective variety  $X$ , we denote by  $\bar{\kappa}(X)$  the logarithmic Kodaira dimension. So when  $X$  is compact, then  $\bar{\kappa}(X)$  is just the usual one, and when  $X$  is non-compact, we let  $\bar{X}$  be a compactification with  $D = \bar{X} \setminus X$  a simple normal crossing divisor and then  $\bar{\kappa}(X) = \bar{\kappa}(\bar{X}, K_{\bar{X}} + D)$  as defined by Iitaka, which is independent of the choice of the compactification.

## §3. Proof of the Theorem.

Let  $X$  denote a smooth affine surface which admits a proper morphism  $f : X \rightarrow X$  with degree  $f > 1$ . Our proof of the classification splits into many cases (see §3.1 ~ §3.7 below).

We will begin with the easier case when  $\bar{\kappa}(X) \geq 0$ . Then by a basic result of Iitaka [10] the map  $f$  is étale. Since  $f$  is also proper by assumption it follows by covering space theory that  $\chi(X) = \deg f \cdot \chi(X)$ . Hence  $\chi(X) = 0$ .

### 3.1. Suppose that $\bar{\kappa}(X) = 2$ .

We claim that this case cannot occur. As a corollary of an inequality of Miyaoka-Yau type by R. Kobayashi, S. Nakamura and F. Sakai it follows that in this case  $\chi(X) > 0$  ([16]). Hence  $f$  cannot exist in this case.

### 3.2. Suppose that $\bar{\kappa}(X) = 1$ .

Since  $X$  is affine, by a basic result due to Kawamata there is a  $\mathbf{C}^*$ -fibration  $\varphi : X \rightarrow B$  ([15], Chapter II, §2). Using Suzuki's formula we can calculate  $\chi(X)$  in terms of  $\chi(B)$  and  $\chi(F_i)$ , where  $F_i$  are the singular fibers of  $\varphi$  ([20] and [22]). Since  $\chi(X) = 0$ , it follows from this formula that every fiber of  $\varphi$  is isomorphic to  $\mathbf{C}^*$ , if taken with reduced structure. Let  $m_1 F_1, m_2 F_2, \dots, m_r F_r$  be all the multiple fibers with multiplicities  $m_i$  of  $\varphi$ .

Suppose first that  $r = 0$ . The  $\mathbf{C}^*$ -fibration  $\varphi$  may not be Zariski locally-trivial but there is a 2-sheeted étale covering  $\Delta \rightarrow B$  such that the fiber product  $\Delta \times_B X$  is a Zariski locally-trivial fibration. Then it is easy to see that  $\bar{\kappa}(X) = 1$  implies that  $\bar{\kappa}(B) = 1$  [11]. Hence either  $B$  is a non-rational curve or a rational curve with at least three places at infinity. It follows by Lüroth's theorem that  $f$  maps any fiber of  $\varphi$  onto another fiber of  $\varphi$ . This induces an étale, proper self-map  $f_0 : B \rightarrow B$ . Now  $\chi(B) < 0$ , since  $\bar{\kappa}(B) = 1$ . Hence  $f_0$  is an automorphism which must have finite order since  $B$  is of general type. Hence  $f^N$  induces an identity on  $B$  for some  $N > 0$ . Let  $g := f^N$ . In view of Claim 3.2a below we will assume that  $\varphi$  is itself locally trivial. We can find an open embedding  $X \subset W$  such that  $W$  has a  $\mathbf{P}^1$ -fibration  $\bar{\varphi} : W \rightarrow B$  and  $W \setminus X$  is a disjoint union of two cross-sections  $S_1, S_2$  of  $\bar{\varphi}$ . The map  $g$  extends to  $W$  mapping every fiber of  $\bar{\varphi}$  to itself. By considering  $g^2$ , if necessary, we can assume that  $S_1, S_2$  are pointwise fixed.

**Claim 3.2a.** There is a finite étale cover  $\Delta' \rightarrow B$  such that  $X \times_B \Delta' \cong$

$\Delta' \times \mathbf{C}^*$ .

*Proof of the claim.* We may assume that  $\Delta = B$ . Restricted to each fiber of  $\varphi$ , the map  $g$  has the form  $t \rightarrow at^m$ , where  $a$  is a non-zero complex number and  $m$  is an integer  $\geq 2$  (since  $\deg g > 1$ ). We can cover  $B$  by open subsets  $U_1, U_2$  such that  $\varphi$  is trivial on each  $U_i$ . Let  $\varphi$  be obtained by patching  $U_1 \times \mathbf{C}^*$ ,  $U_2 \times \mathbf{C}^*$  in  $(U_1 \cap U_2) \times \mathbf{C}^*$  by  $(z, t) \sim (z, \eta(z)t)$ , where  $\eta$  is a regular invertible function on  $U_1 \cap U_2$ . The map  $g$  can be described on  $U_i \times \mathbf{C}^*$  by  $g(z, t) = (z, \alpha_i(z)t^m)$ , where  $\alpha_i$  is a unit on  $U_i$ . Patching gives the relation  $\alpha_1\eta = \alpha_2\eta^m$ , i.e.  $\alpha_1/\alpha_2 = \eta^{m-1}$ . Hence the  $\mathbf{C}^*$ -bundle  $\varphi$  with transition function  $\eta$  is torsion of order  $m - 1$ . It follows that there is an étale cover  $\Delta' \rightarrow B$  of degree  $m - 1$  with the required property. This proves the claim.

Now we know that  $X \cong (\Delta' \times \mathbf{C}^*)/H$ , the action of  $H$  being fixed point free. Here  $H$  is a finite group of degree  $m$  or  $2m$ .

Next assume that  $r > 0$ . By the solution of Fenchel's conjecture due to Fox and Bundgaard-Nielson (see [1], [2], [4]), there exists a Galois covering  $\tilde{B} \rightarrow B$ , ramified precisely over  $\varphi(F_i)$  with ramification index  $m_i$  for  $i = 1, 2, \dots, r$ . Then the normalization  $Y = \overline{X \times_B \tilde{B}}$  of the fiber product  $X \times_B \tilde{B}$  is an étale cover of  $X$ . The induced  $\mathbf{C}^*$ -fibration  $Y \rightarrow \tilde{B}$  has no singular fibers. Hence  $Y \cong (C \times \mathbf{C}^*)/H$  for some étale cover  $C \rightarrow \tilde{B}$  of degree equal to  $|H|$ . By going to a further covering  $C'$  of  $C$ , if necessary, we can assume that  $C' \rightarrow B$  is finite Galois and  $\overline{X \times_B C'}$  is an étale Galois covering of  $X$ . Thus  $X$  is a quotient of a surface of the form  $C' \times \mathbf{C}^*$  by a fixed point free action of a finite group.

This completes the description of  $X$  when  $\bar{\kappa}(X) = 1$ .

**3.3.** Assume now that  $\bar{\kappa}(X) = 0$ .

Since  $\chi(X) = 0$ , we see that  $b_1(X) \neq 0$ . Let  $\mathcal{A}(X)$  be the quasi-Albanese variety of  $X$  and let  $\alpha : X \rightarrow \mathcal{A}(X)$  be the quasi-Albanese map ([12]). There is a short exact sequence:

$$(0) \rightarrow (\mathbf{C}^*)^l \rightarrow \mathcal{A}(X) \rightarrow Alb(V) \rightarrow (0)$$

where  $V$  is a smooth projective compactification of  $X$  and  $Alb(V)$  the Al-

banese variety of  $V$ . Since  $\bar{\kappa}(X) = 0$ , by a result of Kawamata  $\alpha$  is a dominant map with a connected general fiber ([12]).

We assert that if the image of  $X \rightarrow \mathcal{A}(X)$  is a curve then  $\bar{\kappa}F = 0$  with  $F$  a general fibre of  $\alpha$ , whence  $F \cong \mathbf{C}^*$  because  $X$  is affine. Indeed, the easy addition for log Kodaira dimension implies that  $\bar{\kappa}(F) \geq 0$ . On the other hand, Kawamata also proved  $\bar{\kappa}(X) \geq \bar{\kappa}(F) + \bar{\kappa}(\mathcal{A}(X))$ . So to show the assertion, we have only to show that  $\bar{\kappa}(\mathcal{A}(X)) \geq 0$ . To see this, note that  $\mathcal{A}(X)$  is either isomorphic to the Abelian variety  $Alb(V)$ , or to  $\mathbf{C}^*$  whence  $\bar{\kappa}(\mathcal{A}(X)) \geq 0$ . This also proves the assertion.

If the image of  $\alpha$  is a surface then we consider the composite map  $X \rightarrow \mathcal{A}(X) \rightarrow Alb(V)$ . If the image of this composite map is a curve then again  $X$  has a  $\mathbf{C}^*$ -fibration.

In this case and in the case where the image of  $X \rightarrow \mathcal{A}(X)$  is a curve,  $X$  has a  $\mathbf{C}^*$ -fibration. Then we argue as in the previous case. From  $\chi(X) = 0$  we see that the  $\mathbf{C}^*$ -fibration has at most multiple  $\mathbf{C}^*$ 's as singular fibers. We can again conclude that  $X$  is a quotient of  $\Delta \times \mathbf{C}^*$  by a finite fixed point-free automorphism group for a suitable curve  $\Delta$ .

Assume next that the image of  $X \rightarrow Alb(V)$  is a surface. We shall show that this case does not occur. Indeed, note that the image of  $\alpha$  is also a surface and by [12]  $\alpha$  is birational. Thus  $2 = \dim X \geq \dim \mathcal{A}(X) \geq \dim Alb(V) \geq 2$ , whence in the above exact sequence, one has  $l = 0$ . So  $\mathcal{A}(X) = Alb(X)$ . Thus the map  $X \rightarrow Alb(V)$  is birational. There is a curve  $C$  in  $X$  (which may be reducible or empty) such that  $C$  maps to finitely many (smooth) points in  $Alb(V)$  such that  $X \setminus C$  is an open subset of  $Alb(V)$ . Then  $\bar{\kappa}(X \setminus C) = \bar{\kappa}(X) = 0$ . But  $X \setminus C$  is an affine subset of the abelian variety  $Alb(V)$  so its complement in  $Alb(V)$  is an ample divisor (see [9]). Hence we see easily that  $\bar{\kappa}(X \setminus C) = 2$ , a contradiction.

Finally, if the map  $X \rightarrow Alb(X)$  is trivial then there is a non-constant map  $X \rightarrow (\mathbf{C}^*)^l$ , and hence a map  $X \rightarrow \mathbf{C}^*$  which must then be a  $\mathbf{C}^*$ -fibration by Kawamata's inequality in [11] and by  $\bar{\kappa}(X) = 0$ . Thus, when  $\bar{\kappa}(X) = 0$ , as in the previous case,  $X \cong Y/G$  for some smooth affine surface  $Y$  which is isomorphic to  $\Delta \times \mathbf{C}^*$  for some curve  $\Delta$  and  $G$  acting fixed point freely on  $Y$ .

**3.4.** Now assume that  $\bar{\kappa}(X) = -\infty$ .

Note that  $X$  has a compactification so that the complement of  $X$  supports an ample divisor (and hence connected) by [9]. By a basic result due to Fujita, Miyanishi and Sugie ([15], Chapter I, §3), there is an  $\mathbf{A}^1$ -fibration  $\varphi : X \rightarrow B$ .

**Proposition 3.4.** Suppose that  $\bar{\kappa}(X) = -\infty$  and  $\varphi : X \rightarrow B$  is an  $\mathbf{A}^1$ -fibration such that  $\bar{\kappa}(B) \geq 0$ . Then every fibre of  $\varphi$  is reduced and irreducible.

We now prove the proposition. So suppose that  $\bar{\kappa}(B) \geq 0$ . Note that  $f$  permutes the fibers of  $\varphi$ .

We first treat the case when some fiber  $F_0$  of  $\varphi$  is not irreducible, though a general fibre is a reduced curve isomorphic to  $\mathbf{A}^1$ .

Then  $f^{-1}(F_0)$  is also not irreducible since  $\varphi$  is surjective. As before there is an induced map  $f_0 : B \rightarrow B$ . By considering  $f^N$  for suitable  $N$  we can assume that  $f_0$  is an identity map and every irreducible component of  $F_0$  is mapped to itself by  $f$ .

In this case we prove a general result which will be useful in later arguments as well.

**Lemma 3.4A.** *There is no proper self-map  $f : X \rightarrow X$  which maps any irreducible component of any fiber of  $\varphi$  to itself.*

*Proof.* For an irreducible component  $C$  of  $F_0$ , let  $f^*(C) = aC$ . Since the induced map on  $B$  is identity we easily deduce that  $a = 1$ . Since  $X$  is affine the fiber  $F_0$  is a disjoint union of curves isomorphic to  $\mathbf{A}^1$ . It follows that no irreducible component of any fiber of  $\varphi$  is ramified for the map  $f$ . Let  $X_0$  be obtained from  $X$  as follows. For any reducible fiber of  $\varphi$  other than  $F_0$  we remove all but one irreducible components. If  $F_0$  is non-reduced then we remove all the irreducible components of  $F_0$  except for one non-reduced irreducible component. If  $F_0$  is reduced then we remove all but two irreducible components of  $F_0$ . Using the solution of Fenchel's conjecture by Bundagaard-Nielsen, Fox [1], [4] and noting that  $B \neq \mathbf{P}^1$ , we can construct a finite Galois ramified cover  $\tilde{B} \rightarrow B$  with prescribed ramification such that the normalization of the fiber product  $X_0 \times_B \tilde{B}$ , say  $\tilde{X}$ , is a finite étale cover of  $X_0$ . By the universal property of fiber products it follows that  $f$  lifts to a finite self-map  $\tilde{f} : \tilde{X} \rightarrow \tilde{X}$ . The induced  $\mathbf{A}^1$ -fibration on  $\tilde{X}$  has all fibers

reduced and at least one fiber not irreducible. From this observation it now suffices to assume that  $X_0 = \widetilde{X}$ .

Let  $C_1, C_2$  be distinct irreducible components of  $F_0$ . Denote by  $X_0$  the affine surface obtained from  $X$  by removing all but one irreducible components of all the fibers of  $\varphi$ , other than  $F_0$ , and all the irreducible components of  $F_0$ , other than  $C_1, C_2$ . Then there is an induced proper self-map  $f_0 : X_0 \rightarrow X_0$  which maps every irreducible fiber-component to itself. Every fiber of the  $\mathbf{A}^1$ -fibration is reduced. Let  $S$  be the affine scheme over the power series ring in one variable  $k[[t]]$  obtained from  $X_0$  with an  $\mathbf{A}^1$ -fibration over  $k[[t]]$  whose special fiber is reduced with irreducible components  $C_1, C_2$ . To complete the proof of Lemma 3.4A, it suffices to prove that there is no finite self-map  $f : S \rightarrow S$  of degree  $> 1$ .

For this we use an idea from Fieseler's paper ([3], §1). Let  $U := \operatorname{Spec} k[[t]]$ ,  $U^* := U - \{(t)\}$ . Since the special fiber is reduced it is clear that  $S - C_1 \cong U \times \mathbf{A}^1$ ,  $S - C_2 \cong U \times \mathbf{A}^1$ . Now  $S$  is obtained by patching  $S - C_1$ ,  $S - C_2$  along  $U^* \times \mathbf{A}^1$ . Let the patching map  $p : (S - C_1) - C_2 \rightarrow (S - C_2) - C_1$  be given by  $p(t, y) = (t, \alpha(t)y + \beta(t))$ , where  $\alpha, \beta \in k((t))$  and  $\alpha \neq 0$ . The self-map  $f$  restricted to  $S - C_1$  is given by  $f(t, y) = (t, a_0(t)y^m + a_1(t)y^{m-1} + \dots + a_m(t))$ . Each  $a_i \in k[[t]]$ . Since  $f$  has degree  $m$  restricted to each fiber and also restricted to  $C_1, C_2$ , our  $a_0$  is a unit. Similarly,  $f$  restricted to  $S - C_2$  is given by  $f(t, y) = (t, b_0y^m + \dots + b_m)$ . Using the patching we have  $p \circ f = f \circ p$ . This means

$$\beta + \alpha(a_0y^m + a_1y^{m-1} + \dots + a_m) = b_0(\alpha y + \beta)^m + b_1(\alpha y + \beta)^{m-1} + \dots + b_m.$$

Equating the first two leading coefficients, we get:

$$\alpha a_0 = b_0 \alpha^m,$$

$$\alpha a_1 = b_0 m \alpha^{m-1} \beta + b_1 \alpha^{m-1}.$$

Solving them, we obtain:

$$\alpha^{m-1} = a_0/b_0,$$

$$(ma_0)\beta = \alpha a_1 - a_0 b_1/b_0.$$

Since  $a_0$  and  $b_0$  are units, it follows that  $\alpha, \beta \in k[[t]]$ . Since the special fiber is a disjoint union of  $C_1, C_2$  the map  $p$  cannot extend to a morphism



$S - C_1 \rightarrow S - C_2$ . Therefore  $\alpha, \beta$  cannot be both in  $k[[t]]$ . This contradiction completes the proof of Lemma 3.4A.

Now assume that every fiber of  $\varphi$  is irreducible.

If further some fiber is non-reduced then by going to a suitable ramified cover  $\Delta \rightarrow B$  we see that the normalization of the fiber product  $Y := \overline{X \times_B \Delta}$  is an étale cover of  $X$  and has an  $\mathbf{A}^1$ -fibration over  $\Delta$  with all reduced fibers and at least one fiber which is not irreducible. By construction  $f$  lifts to a proper self-map  $f' : Y \rightarrow Y$  of degree  $> 1$ . By the previous case such a map cannot exist. This completes the proof of the proposition.

Now assume that every fiber of  $\varphi$  is reduced and irreducible. We consider the cases  $\bar{\kappa}(B) = 1, 0$  separately.

Consider first the case  $\bar{\kappa}(B) = 1$ .

The induced map  $f_0 : B \rightarrow B$  is an étale, finite map, hence an automorphism of finite order. By taking  $f^N$  for suitable  $N \geq 1$  we will assume that  $f_0$  is identity.

In this case we will prove that there is a finite étale cover  $\tilde{B} \rightarrow B$  such that the fiber product  $\tilde{B} \times_B X$  is a trivial  $\mathbf{A}^1$ -bundle over  $\tilde{B}$ . As above we can embed  $X$  into a smooth surface  $V$  with a  $\mathbf{P}^1$ -fibration  $\Phi : V \rightarrow B$  extending  $\varphi$  such that  $V - X$  is a cross-section  $S$  of  $\Phi$ . It is easy to see that  $f$  extends to a self-map  $V \rightarrow V$  which we write again by  $f$  for simplicity. Now  $f$  maps any point in  $S$  to itself. We argue as in the proof of claim 3.2a. Let  $B = U_1 \cup U_2$  be an open cover such that the  $\mathbf{A}^1$ -fibration on each  $U_i$  is trivial. Let the patching be given on  $U_1 \cap U_2$  be  $(z, t) \sim (z, \eta(z)t + \xi(z))$ , where  $\eta$  is a nowhere vanishing regular function on  $U_1 \cap U_2$  and  $\xi$  is regular on  $U_1 \cap U_2$ . Let  $f$  over  $U_1$  be given by  $f(z, t) = (z, a_0(z)t^d + a_1(z)t^{d-1} + \dots + a_d(z))$  and over  $U_2$  by  $f(z, t) = (z, b_0(z)t^d + b_1(z)t^{d-1} + \dots + b_d(z))$ . Here  $a_i, b_j$  are regular functions on  $U_1, U_2$  respectively and  $a_0, b_0$  are nowhere zero on  $U_1, U_2$  respectively. Using patching we get on  $U_1 \cap U_2$  the following

$$\eta(z)(a_0 t^d + a_1 t^{d-1} + \dots + a_d) + \xi = b_0(\eta t + \xi)^d + b_1(\eta t + \xi)^{d-1} + \dots + b_d.$$

Comparing the coefficients of  $t^d$  on both sides we get  $\eta a_0 = b_0 \eta^d$ . Hence  $\eta^{d-1} = a_0/b_0$ , showing that there is a torsion line bundle on  $B$  whose order

divides  $d - 1$ . Consider the étale cover  $\widetilde{U}_1$  of  $U_1$  obtained by adjoining  $\widetilde{a}_0 := a_0^{1/(d-1)}$  to the coordinate ring of  $U_1$  and similarly let  $\widetilde{U}_2$  be obtained by adjoining  $\widetilde{b}_0 := b_0^{1/(d-1)}$  to the coordinate ring of  $U_2$ . Since  $\widetilde{a}_0/\widetilde{b}_0 = \eta$  these patch to give an étale cover  $\widetilde{B}$  of  $B$ . The self-map  $f$  extends to a proper self-map  $\widetilde{f} : \widetilde{X} \rightarrow \widetilde{X}$ , where  $\widetilde{X} = X \times_B \widetilde{B}$ . We will show that the pull-back  $\mathbf{A}^1$ -fibration  $\widetilde{X}$  is a trivial  $\mathbf{A}^1$ -bundle over  $\widetilde{B}$ . It is easy to see that the patching on  $\widetilde{U}_1 \cap \widetilde{U}_2$  can be assumed to be of the form  $\widetilde{p}(z, t) = (z, t + \xi)$ . Writing the self-map  $\widetilde{f}$  on  $\widetilde{U}_i$  as above we can assume by our construction that the coefficients  $\widetilde{a}_0 = \widetilde{b}_0$ . Comparing the coefficients of  $t^{d-1}$  we get  $\widetilde{a}_1 = d\widetilde{b}_0\xi + \widetilde{b}_1$ . Hence  $\xi = (\widetilde{a}_1 - \widetilde{b}_1)/d\widetilde{b}_0$ . Since  $\widetilde{a}_0 = \widetilde{b}_0$ , the function  $\widetilde{b}_0$  is a nowhere vanishing function on whole of  $\widetilde{B}$ . By changing the notation,  $\xi$  is a difference of two regular functions  $\widetilde{a}_1, \widetilde{b}_1$  on  $\widetilde{U}_1, \widetilde{U}_2$  respectively. From this it is easy to deduce that the  $\mathbf{A}^1$ -fibration on  $\widetilde{B}$  is trivial.

Hence we have shown that  $X$  is a quotient of a product  $\widetilde{B} \times \mathbf{A}^1$  by a finite cyclic group acting fixed point freely.

Consider next the case  $\overline{\kappa}(B) = 0$ . If  $B \cong \mathbf{C}^*$  then  $X \cong \mathbf{A}^1 \times \mathbf{C}^*$ . In this case  $X$  clearly has proper self-map of arbitrary degree.

Suppose that  $B$  is an elliptic curve. Clearly  $f$  permutes the fibers of  $\varphi$ , thus inducing a self-map  $f_0 : B \rightarrow B$ . We can choose a relatively minimal ruled surface  $V$  as a compactification of  $X$  so that  $V \setminus X$  equals the cross-section  $S$  at "infinity" of the unique ruling  $V \rightarrow B$ . One sees easily that  $f : X \rightarrow X$  extends to a self map of  $V$ , also denoted by  $f : V \rightarrow V$  (here the irrationality of the base  $B$  is essentially used). We shall show that there is no such  $f$  of degree  $> 1$ .

**Lemma 3.4B.** Suppose that  $\overline{\kappa}(X) = -\infty$  and the base curve  $B$  of the  $\mathbf{A}^1$ -fibration  $\varphi : X \rightarrow B$  (each fibre of which is an irreducible and reduced  $\mathbf{A}^1$ ) is a nonsingular elliptic curve. Then there is no self map  $f : X \rightarrow X$  of degree  $> 1$ .

*Proof.* Suppose the contrary that  $f$  is a self map of  $X$  of degree  $> 1$ . We use the notation above:  $f : V \rightarrow V$ ,  $S = V \setminus X$ , etc. Write  $f^*S = eS$  with  $e$  the ramification index. Let  $e'$  be the index of function fields extension  $|\mathbf{C}(S) : \mathbf{C}(f(S))|$ . Then  $\deg(f) = ee'$ . Since  $ee'S^2 = (f^*S)^2 = (eS)^2 = e^2S^2$  and since  $S$  is ample (so  $S^2 > 0$ ), we have  $e = e'$ .

We assert that  $f : X \rightarrow X$  is étale. Suppose the contrary that this  $f$  is ramified along a divisor  $D^0$  and let  $D$  be the closure in  $V$  (the purity of branch locus over regular ring, is used). The ampleness of  $S$  implies that  $S \cap D$  contains a point  $P$ . Thus  $f|_S : S \rightarrow S$  is ramified at  $P$ . This is impossible because  $S$  is an elliptic curve. So the assertion is true and  $f : X \rightarrow X$  is étale. In particular, for every fibre  $F$  on  $X$ , the map  $f : F \rightarrow f(F)$  is étale and hence an isomorphism because  $F \cong \mathbf{A}^1$  is simply connected. Since  $S$  is ramified the point at infinity on any  $F$  is ramified. This is a contradiction. Hence  $\deg f = 1$ . This proves the lemma.

Now we are left with the case  $\bar{\kappa}(B) = -\infty$ . Then  $B = \mathbf{A}^1$  or  $\mathbf{P}^1$ .

**Lemma 3.5.** It is impossible that  $B = \mathbf{A}^1$  and the fundamental group  $\pi_1(X)$  is infinite.

*Proof.* Suppose the contrary that  $B = \mathbf{A}^1$  and  $\pi_1(X)$  is infinite. We claim that the first Betti number  $b_1(X) = 0$ . This follows from the exact sequence for homology groups with rational coefficients ([20]):

$$H_1(\mathbf{A}^1) \rightarrow H_1(X) \rightarrow H_1(B) \rightarrow (0).$$

It follows that  $\chi(X) > 0$ . Since  $\pi_1(X)$  is infinite we see that  $\varphi$  has at least two multiple fibers (otherwise,  $\pi_1(X)$  is finite cyclic). Let  $\Delta \rightarrow B$  be a suitable ramified Galois covering such that  $Y := \overline{X \times_B \Delta} \rightarrow X$  is étale and  $\psi : Y \rightarrow \Delta$  is an  $\mathbf{A}^1$ -fibration with all reduced fibers. Now  $\bar{\kappa}(\Delta) \geq 0$ .

*Claim.*  $f$  extends to a finite map  $g : Y \rightarrow Y$ .

For this we will show that the induced homomorphism  $f_* : \pi_1(X) \rightarrow \pi_1(X)$  is an isomorphism. To see this, first we know that this image always has finite index. This observation goes back to Serre (cf. [19], Lemma 1.5). Let  $Z \rightarrow X$  be the finite étale cover such that  $\pi_1(Z)$  is equal to the above subgroup of finite index. By covering space theory,  $f$  extends to a morphism  $f' : X \rightarrow Z$ . Let  $d$  be the index of  $\pi_1(Z)$  in  $\pi_1(X)$ . Then  $\chi(Z) = d\chi(X)$ . Since  $X$  dominates  $Z$ , by Lemma 1.5 in [19] we know that  $0 = b_1(X) = b_1(Z)$ . Since  $f'$  is also a finite map it is known that  $b_2(X) \geq b_2(Z)$  [6]. Now  $\chi(Z) = d\chi(X) \geq d\chi(Z)$ , we have a contradiction if  $d \geq 2$ . This proves the assertion that  $f_*$  is onto.

We claim that, in fact,  $f_*$  is an isomorphism.

To see this, let  $m_1F_1, m_2F_2, \dots, m_rF_r$  be the multiple fibers of  $\varphi$ . A slight extension of Lemma 1.5 in [19] shows that we have an isomorphism (using the fact that a general fiber of  $\varphi$  is  $\mathbf{A}^1$ )

$$\pi_1(X) \cong \langle e_1, e_2, \dots, e_r | e_1^{m_1} = 1 = \dots = e_r^{m_r} \rangle$$

This group is known to be residually finite, i.e. the intersection of all its subgroups of finite index is trivial ([8]). By a result of Malcev any finitely generated residually finite group  $G$  has the property that any surjective homomorphism  $G \rightarrow G$  is an isomorphism ([13]). Hence the surjection  $f_*$  is an isomorphism.

Now by covering space theory  $f$  extends to a finite self map  $g : Y \rightarrow Y$ . This proves the claim.

At least two fibers of  $\psi$  (over  $b_1, b_2 \in \Delta$  say) are not irreducible. Note that  $g$  induces an automorphism  $g|_{\Delta}$  on the base  $\Delta$ . A power  $h = g^s$  induces  $h|_{\Delta}$  fixing  $b_1, b_2$ . If the affine curve  $\Delta$  has  $\bar{\kappa}(\Delta) \geq 0$ , we see that a higher power  $g^{sn}$  induces the identity  $:g^{sn}|_{\Delta} = id$ ; if  $\bar{\kappa}(\Delta) = -\infty$ , then  $\Delta = \mathbf{A}^1 \subset \mathbf{P}^1 = \Delta \cup \{\infty\}$  and  $g^s$  induces an automorphism of  $\mathbf{P}^1$  fixing  $b_1, b_2, \infty$  (so  $g^s|_{\mathbf{P}^1} = id$ ), whence  $g^s$  stabilizes every fibre of  $\varphi$ . Now we can apply the argument in the proof of Lemma 3.4A and conclude that  $g^{sn}$  cannot exist. Hence  $f$  also does not exist. This proves the lemma.

**Lemma 3.6.** It is impossible that  $B = \mathbf{P}^1$  and the fundamental group  $\pi_1(X)$  is infinite.

The proof in this case is similar to the above case. Indeed, if  $\pi_1(X)$  is infinite, then  $\varphi$  has at least three multiple fibers and in the case of three multiple fibers the multiplicities do not form a Platonic triple. Now the rest of the argument is very similar to the previous case.

This completes the proof of the Theorem and also its Corollary.

**Remark 3.7.** Now we are left with the case  $\bar{\kappa}(X) = -\infty$ ,  $\bar{\kappa}(B) = -\infty$  and  $\pi_1(X)$  is finite. This case appears to be much harder.

For example, let  $X$  be the affine surface in  $\mathbf{C}^3$  defined by  $\{XY - Z^2 = 1\}$ . It is well-known that  $X \cong \mathbf{P}^1 \times \mathbf{P}^1 - \text{Diagonal}$ . Clearly,  $X$  has an  $\mathbf{A}^1$ -

fibration over  $\mathbf{P}^1$  which is Zariski-locally trivial.  $X$  is simply-connected and  $\overline{\kappa}(X) = -\infty$ . The fibration  $X \rightarrow \mathbf{A}^1$  given by the function  $x$  is an  $\mathbf{A}^1$  fibration with all reduced fibers. It is not clear even in this case if  $X$  has a finite self-morphism of degree  $> 1$ .

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